ON THE q-EULER NUMBERS AND POLYNOMIALS WITH WEIGHT 0

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Abstract The purpose of this paper is to investigate some properties of q-Euler numbers and polynomials with weight 0. From those q-Euler numbers with weight 0, we derive some identies on the q-Euler numbers and polynomials with weight 0.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolutely value is defined by $|x|_p = 1/p^r$ where $x = p^r s/t$ with $(p,s) = (p,t) = (s,t) = 1$ and $r \in \mathbb{Q}$. In this paper, we assume that $\alpha \in \mathbb{Q}$ and $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. As well known definition, the Euler polynomials are defined by

$$\frac{2}{e^t + 1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$ (see [1-15]).

In this special case, $x = 0, E_n(0) = E_n$ are called the n-th Euler numbers (see [1]). Recently, the q-Euler numbers with weight α are defined by

$$\tilde{E}_{0,q}^{(\alpha)} = 1, \quad \text{and} \quad q(q^\alpha \tilde{E}_q^{(\alpha)} + 1)^n + \tilde{E}_{n,q}^{(\alpha)} = 0 \ \text{ if } \ n > 0, \tag{1}$$

with the usual convention about replacing $\left(\tilde{E}_q^{(\alpha)}\right)^n$ by $\tilde{E}_{n,q}^{(\alpha)}$ (see [3,12]). The q-number of x is defined by $[x]_q = \frac{1-q^x}{1-q}$ (see [1-15]). Note that $\lim_{q\to1}[x]_q = x$. Let us define the notation of q-Euler numbers with weight 0 as $\tilde{E}_{n,q}^{(0)} = \tilde{E}_{n,q}$. The purpose of this paper is to investigate some interesting identities on the q-Euler numbers with weight 0.

2. ON THE EXTENDED q-EULER NUMBERS OF HIGHER-ORDER WITH WEIGHT 0

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows :

$$
\begin{aligned}
I_q(f) &= \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) \\
&= \lim_{N\to\infty} \frac{[2]_q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \ \text{(see [1-12])}.
\end{aligned}
\tag{2}
$$

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By (2), we get

$$q^n I_q(f_n) + (-1)^{n-1} I_q(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l) q^l, \tag{3}$$

where $f_n(x) = f(x+n)$ and $n \in \mathbb{N}$ (see [4, 5]).

By (1), (2) and (3), we see that

$$\int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_{-q}(x) = \tilde{E}_{n,q}^{(\alpha)} = \frac{[2]_q}{(1-q)^n [\alpha]_q^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1+q^{\alpha l+1}}. \tag{4}$$

In the special case, $n = 1$, we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^t+1} = \frac{1+q^{-1}}{e^t+q^{-1}} = \sum_{n=0}^{\infty} H_n(-q^{-1}) \frac{t^n}{n!}, \tag{5}$$

where $H_n(-q^{-1})$ are the n-th Frobenius-Euler numbers. From (5), we note that the q-Euler numbers with weight 0 are given by

$$\tilde{E}_{n,q} = \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = H_n(-q^{-1}), \text{ for } n \in \mathbb{Z}_+. \tag{6}$$

Therefore, by (6), we obtain the following theorem.

Theorem 1. *For $n \in \mathbb{Z}_+$, we have*

$$\tilde{E}_{n,q} = H_n(-q^{-1}),$$

where $H_n(-q^{-1})$ are called the n-th Frobenius-Euler numbers.

Let us define the generating function of the q-Euler numbers with weight 0 as follows:

$$\tilde{F}_q(t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q} \frac{t^n}{n!}. \tag{7}$$

Then, by (4) and (7), we get

$$\tilde{F}_q(t) = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m e^{mt} = \frac{1+q}{qe^t+1}. \tag{8}$$

Now we define the q-Euler polynomials with weight 0 as follows:

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!} = \frac{1+q}{qe^t+1} e^{xt}. \tag{9}$$

Thus, (5) and (9), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \frac{1+q}{qe^t+1} e^{xt} = \sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!}. \tag{10}$$

From (10), we have

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q}(x) \frac{t^n}{n!} = \left(\frac{1+q^{-1}}{e^t+q^{-1}} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(-q^{-1}, x) \frac{t^n}{n!}, \tag{11}$$

where $H_n(-q^{-1}, x)$ are called the n-th Frobenius-Euler polynomials (see[9]).

Therefore, by (11), we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{Z}_+$, we have*

$$\tilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(x) = H_n(-q^{-1}, x),$$

where $H_n(-q^{-1}, x)$ are called the n-th Frobenius-Euler polynomials.

From (3) and Theorem 2, we note that

$$q^n H_m(-q^{-1}, n) + H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l, \tag{12}$$

where $n \in \mathbb{N}$ with $n \equiv 1 \pmod 2$.

Therefore, by (12), we obtain the following corollary.

Corollary 3. *For $n \in \mathbb{N}$, with $n \equiv 1 \pmod 2$ and $m \in \mathbb{Z}_+$, we have*

$$q^n H_m(-q^{-1}, n) + H_m(-q^{-1}) = [2]_q \sum_{l=0}^{n-1} (-1)^l l^m q^l.$$

In particular, $q = 1$, we get $E_m(n) + E_m = 2 \sum_{l=0}^{n-1} (-1)^l l^m$, where E_m and $E_m(n)$ are called the m-th Euler numbers and polynomials which are defined by

$$\frac{2}{e^t + 1} = \sum_{m=0}^{\infty} E_m \frac{t^m}{m!} \quad \text{and} \quad \frac{2}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} E_m(x) \frac{t^m}{m!}.$$

By (3), we easily see that

$$q \int_{\mathbb{Z}_p} f(x+1) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2]_q f(0). \tag{13}$$

Thus, by (13), we get

$$
\begin{aligned}
[2]_q &= q \int_{\mathbb{Z}_p} e^{(x+1)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) \\
&= \sum_{n=0}^{\infty} \left(q \int_{\mathbb{Z}_p} (x+1)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(q H_n(-q^{-1}, 1) + H_n(-q^{-1}) \right) \frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by (13), we obtain the following theorem.

Theorem 4. *For $n \in \mathbb{Z}_+$, we have*

$$q H_n(-q^{-1}, 1) + H_n(-q^{-1}) = \begin{cases} 1 + q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

where $H_n(-q^{-1}, x)$ are called the n-th Frobenius-Euler polynomials and $H_n(-q^{-1})$ are called the n-th Frobenius-Euler numbers. In particular, $q = 1$, we have

$$E_n(1) + E_n = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases}$$

where E_n are called the n-th Euler numbers.

From (6) and Theorem 2, we note that

$$
\begin{aligned}
\tilde{E}_{n,q}(x) &= \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) \\
&= \sum_{l=0}^{n} \binom{n}{l} \int_{\mathbb{Z}_p} y^l d\mu_{-q}(y) x^{n-l} \\
&= \sum_{l=0}^{n} \binom{n}{l} \tilde{E}_{n,q} x^{n-l} \\
&= \left(x + \tilde{E}_q\right)^n,
\end{aligned}
\tag{14}
$$

where the usual convention about replacing $(\tilde{E}_q)^l$ by $\tilde{E}_{l,q}$. By Theorem 2 and Theorem 4, we get

$$
q\tilde{E}_{n,q}(1) + \tilde{E}_{n,q} =
\begin{cases}
[2]_q, & \text{if } n = 0, \\
0, & \text{if } n > 0.
\end{cases}
\tag{15}
$$

From (14) and (15), we have

$$
q\left(\tilde{E}_q + 1\right)^n + \tilde{E}_{n,q} =
\begin{cases}
[2]_q, & \text{if } n = 0, \\
0, & \text{if } n > 0.
\end{cases}
\tag{16}
$$

For $n \in \mathbb{N}$, by (14) and (16), we have

$$
\begin{aligned}
q^2 \tilde{E}_{n,q}(2) &= q^2 \left(\tilde{E}_q + 1 + 1\right)^n \\
&= q^2 \sum_{l=1}^{n} \binom{n}{l} \left(\tilde{E}_q + 1\right)^l + q\left(1 + q - \tilde{E}_{0,q}\right) \\
&= q + q^2 - q \sum_{l=0}^{n} \binom{n}{l} \tilde{E}_{l,q} \\
&= q + q^2 - q\left(\tilde{E}_q + 1\right)^n \\
&= q + q^2 + \tilde{E}_{n,q}.
\end{aligned}
\tag{17}
$$

Therefore, by (17), we obtain the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, we have*

$$
q^2 \tilde{E}_{n,q}(2) = q + q^2 + \tilde{E}_{n,q}.
$$

For $n \in \mathbb{Z}_+$, we have

$$
\begin{aligned}
\tilde{E}_{n,q^{-1}}(1-x) &= \int_{\mathbb{Z}_p} (1 - x + x_1)^n d\mu_{-q^{-1}}(x_1) \\
&= (-1)^n \int_{\mathbb{Z}_p} (x_1 + x)^n d\mu_{-q}(x_1) \\
&= (-1)^n \tilde{E}_{n,q}(x).
\end{aligned}
\tag{18}
$$

Therefore, by (18), we obtain the following theorem.

Theorem 6. *For $n \in \mathbb{Z}_+$, we have*

$$\tilde{E}_{n,q^{-1}}(1-x) = (-1)^n \tilde{E}_{n,q}(x).$$

From (14), we have

$$
\begin{aligned}
\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-q}(x) &= (-1)^n \int_{\mathbb{Z}_p} (x-1)^n d\mu_{-q}(x) \\
&= (-1)^n \tilde{E}_{n,q}(-1).
\end{aligned}
\tag{19}
$$

By Therorem 6 and (19), we get

$$
\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-q}(x) = \tilde{E}_{n,q^{-1}}(2) = 1 + q + q^2 \tilde{E}_{n,q^{-1}} \quad \text{if} \quad n > 0.
\tag{20}
$$

Therefore, by (20), we obtain the following theorem.

Theorem 7. *For $n \in \mathbb{N}$, we have*

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-q}(x) = 1 + q + q^2 \tilde{E}_{n,q^{-1}}.$$

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, p-adic analogue of Bernstein operator of order n for f is given by

$$
\begin{aligned}
\mathbb{B}_n(f|x) &= \sum_{k=0}^{n} B_{k,n}(x) f\left(\frac{k}{n}\right) \\
&= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k},
\end{aligned}
\tag{21}
$$

where $n, k \in \mathbb{Z}_+$ (see [1,6,7]).

For $n, k \in \mathbb{Z}_+$, p-adic Bernstein polynomials of degree n is defined by

$$
B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in \mathbb{Z}_p \quad \text{(see [1,6,7])}.
\tag{22}
$$

Let us take the fermionic p-adic q-integral on $\mathbb{Z}_p$ for one Bernstein polynomials in (22) as follows:

$$
\begin{aligned}
\int_{\mathbb{Z}_p} B_{k,n}(x) d\mu_{-q}(x) &= \binom{n}{k} \int_{\mathbb{Z}_p} x^k (1-x)^{n-k} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+l} d\mu_{-q}(x) \\
&= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l \tilde{E}_{k+l,q}
\end{aligned}
\tag{23}
$$

By simple calculation, we easily get

$$\int_{\mathbb{Z}_p} B_{k,n}(x)d\mu_{-q}(x) = \int_{\mathbb{Z}_p} B_{n-k,n}(1-x)d\mu_{-q}(x)$$

$$= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\int_{\mathbb{Z}_p}(1-x)^{n-l}d\mu_{-q}(x)$$

$$= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}\left(1+q+q^2\tilde{E}_{n-l,q^{-1}}\right) \quad (24)$$

$$= \binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}q^2\tilde{E}_{n-l,q^{-1}} \quad \text{if} \quad n > k.$$

Therefore, by (23) and (24), we obtain the following theorem.

Theorem 8. *For $n \in \mathbb{Z}_+$ with $n > k$, we have*

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l\tilde{E}_{k+l,q} = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}q^2\tilde{E}_{n-l,q^{-1}}.$$

In particular, $k = 0$, we get

$$\sum_{l=0}^{n}\binom{n}{l}(-1)^l\tilde{E}_{l,q} = q^2\tilde{E}_{n,q^{-1}}.$$

By Theorem 1 and Theorem 2, we get

$$\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l H_{k+l}(-q^{-1}) = \sum_{l=0}^{k}\binom{k}{l}(-1)^{k+l}q^2 H_{n-l}(-q),$$

where $H_n(-q)$ are called the n-th Frobenius-Euler numbers.

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